Ch 3.4: Proof by cases

Let a statement of the form

$$(P_1 \text{ or } P_2 \text{ or } \cdots \text{ or } P_n) \implies Q$$

be given, where $P_1, P_2, \cdots, P_n$ are cases.

Show that it is equivalent to the following:

$$(P_1 \implies Q) \text{ and } (P_2 \implies Q) \text{ and } \cdots \text{ and } (P_n \implies Q).$$

Thus, we need to prove all the clauses are true.

Example: Prove that if $n \in \mathbb{Z}$, then $n^3 - n$ is even.

Example: Prove "$n^2 \geq n$ for any integer n".

Proof:

1. How many cases do we need to consider?

2. Explicitly state what we want to prove.

3. If we can prove that all cases are true, we can conclude that $n^2 \geq n$ for all integers n.

4. Write the final polished form of your proof.

Exercises

1. For $n \in \mathbb{Z}$, prove that $9n^2 + 3n - 2$ is even.

2. For any integer n, $n^3 + n$ is an even integer.

3. For $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

4. Prove that for all $x \in \mathbb{R}$, $-5 \leq |x+2| - |x-3| \leq 5$.

5. Prove that if x is a real number such that $\dfrac{x^2-1}{x+2} > 0$, then $x > 1$ or $-2 < x < -1$.

More exercise problems for Chapter 3

1. Show that if n is an even integer then either $n = 4k$ or $n = 4k + 2$ for some integer k. (Hint: For n to be even means that $n = 2m$ for some integer m. Consider two possibilities for m.)

2. Prove the following statements by stating and proving the contrapositive

   (a) If n^2 is an odd integer, then n is an odd integer.

   (b) If n^2 is divisible by 4, then n is even.

   (c) Let a and b be nonnegative real numbers. If $a^2 < b^2$, then $a < b$. (Hint: Use the following property of the real numbers: if $a < b$ and $c > 0$, then $ac < bc$.)

   (d) Let a and b be nonnegative real numbers and let $n \in \mathbb{N}$. If $a^n < b^n$, then $a < b$.

3. Use a proof by cases to prove that if $n = m^2$ for some integer m, then $n = 4k$ or $n = 4k + 1$ for some integer k.